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Eigenvalue comparisons for a class of boundary value problems of second order difference equations

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Abstract

We consider the boundary value problem for second order difference equation

$$\Delta(r_{i-1}\Delta y_{i-1}) - b_i y_i + \lambda a_i y_i = 0, \quad 1 \leq i \leq n, \quad y_0 - \tau y_1 = y_{n+1} - \delta y_n = 0.$$

In this study we do not require the positiveness of $\{a_k\}_{k=1}^n$. We focus on the structure of eigenvalues of this problem and comparisons of all eigenvalues as the coefficients $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, $\{r_i\}_{i=0}^n$ and the parameters τ , δ change.

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1. Introduction

A problem of great importance in theoretical physics and applied mathematics is that of determining the values of λ which permit the second order boundary value problem

$$(r(t)y'(t))' - b(t)y(t) + \lambda a(t)y(t) = 0, \tag{1.1a}$$

$$y(0) = y(1) = 0, \tag{1.1b}$$

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to possess a nontrivial solution. The functions $r(t)$, $a(t)$, and $b(t)$ are assumed to be real and continuous over $[0, 1]$. Equally important is the problem (1.1a) with the boundary conditions $y(0) = y'(1) = 0$ or $y'(0) = y(1) = 0$. Since, in general, the differential equation cannot be solved in terms of the elementary transcendents, various approximate techniques must be employed to resolve this problem. If we take the computational approach, then we need to consider the boundary value problem for the second order difference equation

$$\Delta(r_{i-1}\Delta y_{i-1}) - b_i y_i + \lambda a_i y_i = 0, \quad 1 \leq i \leq n, \quad (1.2a)$$

$$y_0 - \tau y_1 = y_{n+1} - \delta y_n = 0, \quad (1.2b)$$

where the forward difference operator Δ is defined as $\Delta y_i = y_{i+1} - y_i$. Here the parameters τ , δ are introduced in order to include various types of boundary conditions in our discussion.

If λ is a number (maybe complex) such that the problem (1.2) has a nontrivial solution $\{y_i\}_{i=0}^{n+1}$, then λ is said to be an eigenvalue of the problem, and the corresponding nontrivial solution $\{y_i\}_{i=0}^{n+1}$ is called an eigenvector of the problem corresponding to λ .

In this paper we will focus on the structure of the eigenvalues of the problem (1.2) as well as the behavior of the eigenvalues, i.e., the comparison of eigenvalues, as the coefficients and the parameters which define the problem change.

The research on the comparison of eigenvalues has been very active recently since the earlier work of Travis [17] where the eigenvalue problem for the higher order boundary value problem

$$\begin{aligned} ((a(x)u^{(n)})^{(n)}) + \lambda(-1)^{n+1}c(x)u &= 0, \\ u(\alpha) = u'(\alpha) = \dots = u^{(n-1)}(\alpha) &= 0, \\ u^{(n)}(\beta) = u^{(n+1)}(\beta) = \dots = u^{(2n-1)}(\beta) &= 0 \end{aligned}$$

was considered and comparison results for the smallest eigenvalues were obtained, by using the theory of u_0 -positive linear operator in a Banach space equipped with a cone of “nonnegative” elements. A representative set of references for these works would be Davis et al. [3], Diaz and Peterson [4], Hankerson and Henderson [6], Hankerson and Peterson [7–9], Henderson and Prasad [10], Kaufmann [13], and Travis [17]. However, in all the aforementioned papers, the focus has been on the smallest eigenvalue.

Spectral properties of second-order discrete Sturm–Liouville problem (1.2) have been of growing interest in recent years. Atkinson [1] and Jirari [11] studied the problem with positive a_i ($1 \leq i \leq n$) by investigating some oscillatory properties of solutions as done in the continuous case. For the extensions to discrete vector Sturm–Liouville problems, see [15,16]. We notice that the problems from real world applications may involve zero or even negative numbers in the sequence $\{a_i\}_{i=1}^n$. To our knowledge, the class of problems with possible nonpositive elements in $\{a_i\}$ has never been discussed in the literature due to its difficulty. Under the assumptions that $r_i > 0$, $0 \leq i \leq n$ and $b_i = 0$, $a_i \geq 0$, $1 \leq i \leq n$, some progress has been made recently in [12] where the structure of the eigenvalues was established and comparisons of all eigenvalues as $\{a_i\}$ changes were obtained.

In this paper, we further extend our earlier results of [12] to a more general setting, allowing some a_i 's to be negative. We will obtain the complete structure of the eigenvalues and describe the monotonic behavior of all eigenvalues as the coefficients $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, $\{r_i\}_{i=0}^n$ and the parameters τ , δ change for the class of problems (1.2) satisfying the assumption (H) below. Throughout we assume that n is a fixed integer and the following condition holds:

- (H) The τ and δ are constants in $[0, 1]$ with $\tau + \delta \neq 2$. The $\{r_i\}_{i=0}^n$, $\{a_i\}_{i=1}^n$, and $\{b_i\}_{i=1}^n$ are finite sequences of real numbers such that $r_i > 0$ for $0 \leq i \leq n$ and $b_i \geq 0$ for $1 \leq i \leq n$. And there is at least one nonzero member of the sequence $\{a_i\}_{i=1}^n$.

2. The structure of the eigenvalues

In what follows we will write $X \geq Y$ for two symmetric $n \times n$ matrices X and Y if $X - Y$ is positive semidefinite. Furthermore, we will write $X > Y$ if $X - Y$ is positive definite. We denote by x^* the conjugate transpose of a vector x , by $\text{Null}(X)$ the null space of a matrix X , and by $\lambda_k(X)$ the k th largest eigenvalue of a real symmetric matrix X .

Note that the problem (1.2) is equivalent to the equation

$$(-G + \lambda A)y = 0, \quad (2.1)$$

where $G = D + B$ and D is a tridiagonal $n \times n$ matrix given by

$$D = \begin{pmatrix} (1-\tau)r_0 + r_1 & -r_1 & 0 & \cdots & 0 & 0 & 0 \\ -r_1 & r_1 + r_2 & -r_2 & \cdots & 0 & 0 & 0 \\ 0 & -r_2 & r_2 + r_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & r_{n-3} + r_{n-2} & -r_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & -r_{n-2} & r_{n-2} + r_{n-1} & -r_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -r_{n-1} & r_{n-1} + (1-\delta)r_n \end{pmatrix},$$

and

$$A = \text{diag}(a_1, a_2, \dots, a_{n-1}, a_n),$$

$$B = \text{diag}(b_1, b_2, \dots, b_{n-1}, b_n),$$

$$y = (y_1, y_2, \dots, y_{n-1}, y_n)^T.$$

Obviously, the matrices D and G depend on $\{r_i\}_{i=0}^n$, $\{b_i\}_{i=1}^n$, τ , and δ . For simplicity, this dependency is not indicated in the notations.

For any n -dimensional real vector $x = (x_1, x_2, \dots, x_n)^T$, we have

$$x^T G x = (1-\tau)r_0 x_1^2 + \sum_{i=1}^{n-1} r_i (x_i - x_{i+1})^2 + (1-\delta)r_n x_n^2 + \sum_{i=1}^n b_i x_i^2. \quad (2.2)$$

Under the assumption (H), it is easy to check that $x^T G x \geq 0$ for any x . Moreover, $x^T G x = 0$ implies $x = 0$. Hence, the symmetric matrix G is positive definite.

Let us first consider the case when each member of the sequence $\{a_i\}_{i=1}^n$ is zero. If there were a number λ and a nonzero vector y satisfying (2.1), then $y^* G y = \lambda y^* A y = 0$ since $A = 0$. It is seen from the positive definiteness of G that $y = 0$ which is impossible. Thus there does not exist any eigenvalues of the problem in this case which justifies our requirement of having at least one nonzero a_{i_0} , $1 \leq i_0 \leq n$ in the assumption (H).

Lemma 2.1. *If λ is an eigenvalue of the problem (1.2) and y is a corresponding eigenvector, then (i) λ is real and nonzero; (ii) $y^* A y \neq 0$; (iii) if $\rho \neq \lambda$ is also an eigenvalue of the problem (1.2) and x is an eigenvector corresponding to ρ , then we have $x^* A y = 0$.*

Proof. First we note that $\lambda y^*Ay = y^*Gy > 0$, since G is positive definite and $y \neq 0$. Hence λ and y^*Ay are both nonzero. We can write

$$\lambda y^*Ay = y^*(\lambda Ay) = y^*Gy = (Gy)^*y = (\lambda Ay)^*y = \bar{\lambda} y^*A^*y = \bar{\lambda} y^*Ay, \quad (2.3)$$

which implies $\lambda = \bar{\lambda}$, i.e., λ is real. Part (iii) follows from

$$\begin{aligned} (\lambda - \rho)x^*Ay &= \lambda x^*Ay - \rho x^*Ay = x^*(\lambda Ay) - (\rho Ax)^*y \\ &= x^*Gy - (Gx)^*y = 0. \end{aligned}$$

The proof is complete. \square

For the positive definite matrix G , there exists a unique lower triangular matrix L such that $LL^T = G$. We note that L is nonsingular. With the help of the Cholesky decomposition, we will convert the eigenvalue problem of the form (2.1) to a regular eigenvalue problem.

Lemma 2.2. *Let $G = LL^T$ be the Cholesky decomposition of G . The eigenvalues of the problem (1.2) are related to those of the matrix $L^{-1}AL^{-T}$ as follows:*

- (a) *If λ is an eigenvalue of the problem (1.2) and y is a corresponding eigenvector, then $1/\lambda$ is an eigenvalue of $L^{-1}AL^{-T}$ and L^Ty is a corresponding eigenvector.*
- (b) *If α is a nonzero eigenvalue of $L^{-1}AL^{-T}$ and y is a corresponding eigenvector, then $1/\alpha$ is an eigenvalue of the problem (1.2) and $L^{-T}y$ is a corresponding eigenvector.*

Proof. (a) If λ is an eigenvalue of the problem (1.2) and y is a corresponding eigenvector, then $\lambda \neq 0$ in view of Lemma 2.1. The equation $\lambda Ay = Gy$ is equivalent to the equation $\lambda Ay = LL^Ty$. Thus, we have $L^{-1}AL^{-T}L^Ty = \frac{1}{\lambda}L^Ty$.

The result in (b) can be proved similarly. The proof is complete. \square

The next result was obtained in [12] when $\tau = 0$ and $b_i = 0$, $1 \leq i \leq n$. This result is still true for the class of problems considered in this paper. For completeness of this paper, a proof is included.

Lemma 2.3. *If λ is an eigenvalue of the problem (1.2), and $y = (y_1, \dots, y_n)^T$ is a corresponding eigenvector, then*

- (a) $y_1 \neq 0$ and $y_n \neq 0$.
- (b) *the nullity of $(-G + \lambda A)$ is 1.*

Proof. For part (a), assume the contrary that either $y_1 = 0$ or $y_n = 0$. Then we can easily deduce a contradiction, $y = 0$, from $(-G + \lambda A)y = 0$. We leave details to the reader.

For part (b), let $x = (x_1, x_2, \dots, x_{n-1}, x_n)^T$ and $y = (y_1, y_2, \dots, y_{n-1}, y_n)^T$ be any two eigenvectors of the problem (2.1) corresponding to λ and define $z = x_1y - y_1x$. Obviously, we have

$$(-G + \lambda A)z = x_1(-G + \lambda A)y - y_1(-G + \lambda A)x = 0,$$

which, together with the fact that $z_1 = 0$, indicates that $z = 0$, that is, $x_1y = y_1x$. Therefore, x and y are linearly dependent thanks to part (a) of this lemma. \square

Theorem 2.4. Let $G = LL^T$ be the Cholesky decomposition of G and let p, q be the number of positive and the number of negative elements in the set $\{a_i\}_{i=1}^n$ respectively. Then there are p distinct positive eigenvalues $\{\lambda_i^+ : i = 1, 2, \dots, p\}$ and q distinct negative eigenvalues $\{\lambda_i^- : i = 1, 2, \dots, q\}$ of the problem (2.1). Moreover,

$$\{1/\lambda_i^+ : i = 1, 2, \dots, p\} \cup \{1/\lambda_i^- : i = 1, 2, \dots, q\}$$

is the set of all nonzero eigenvalues of $L^{-1}AL^{-T}$.

Proof. The assumption (H) implies $p + q \geq 1$. The fact that $L^{-1}AL^{-T}$ is real and symmetric indicates that there exists an orthogonal matrix Q such that

$$Q^T L^{-1} A L^{-T} Q = \text{diag}(\alpha_1, \alpha_2, \dots, \alpha_n), \quad (2.4)$$

where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ are all eigenvalues of $L^{-1}AL^{-T}$. Let $x = L^{-T}Qz$. It is seen from (2.4) that

$$\sum_{i=1}^n \alpha_i z_i^2 = \sum_{i=1}^n a_i x_i^2$$

are two representations of the real quadratic form $x^T A x$. In view of the Law of Inertia for Quadratic Forms [5, Theorem 1, p. 297], we immediately deduce that the number of positive and the number of negative elements in the set $\{\alpha_i : i = 1, 2, \dots, n\}$ are p and q , respectively.

We claim that all the nonzero elements in $\{\alpha_i\}_{i=1}^n$ are distinct. Suppose the contrary that $\alpha_{i_0} = \alpha_{j_0} \neq 0$ for some $1 \leq i_0 < j_0 \leq n$. Denote by e_i the i th column of the identity matrix of order n . In view of (2.4), we have $Q^T L^{-1} A L^{-T} Q e_i = \alpha_i e_i$ for $i = i_0, j_0$, which further implies

$$G(L^{-T}Qe_i) = \frac{1}{\alpha_i} A(L^{-T}Qe_i), \quad i = i_0, j_0.$$

Thus, we have two independent vectors in $\text{Null}(-G + \lambda A)$ for $\lambda = 1/\alpha_{i_0}$, more specifically,

$$L^{-T}Qe_i \in \text{Null}\left(-G + \frac{1}{\alpha_{i_0}}A\right), \quad i = i_0, j_0,$$

contradicting Lemma 2.3. Thus, in view of Lemma 2.2, we see that $\{1/\alpha_i : \alpha_i \neq 0\}$ gives the complete set of eigenvalues of the problem (2.1). Therefore, $\{\lambda_i^+ = 1/\alpha_i : i = 1, 2, \dots, p\}$ and $\{\lambda_i^- = 1/\alpha_{n-i+1} : i = 1, 2, \dots, q\}$ are the sets of all the positive and all the negative eigenvalues of the problem (2.1), respectively. The proof is complete. \square

When $a_i > 0$, $1 \leq i \leq n$, we have $p = n$ and $q = 0$. Therefore, there exist n distinct positive eigenvalues for the problem (1.2) in this case. Note that the existence of n distinct real eigenvalues was obtained in [1] for the problem with two positive sequences $\{r_i\}$ and $\{a_i\}$. The recent paper [12] gave the structure of the problem with positive $\{r_i\}$ and nonnegative $\{a_i\}$ under the assumption that $b_i = 0$, $1 \leq i \leq n$. Theorem 2.4 enhances our earlier results and for the first time, handles the general case which has no restriction on the sign of $\{a_i\}$.

We remark that the result of Theorem 2.4 actually provides a method of calculating all the eigenvalues of problems in the class of boundary value problems of second order difference equations being considered in this paper. This method requires to compute the Cholesky decomposition of G and to employ an existing eigenvalue software package on $L^{-1}AL^{-T}$ for its regular nonzero eigenvalues. The eigenvalues of the original problem can then be easily recovered.

3. The eigenvalue comparison

In this section, we will discuss the dependency of the eigenvalues of the problem (1.2) on its coefficients $\{r_i\}$, $\{a_i\}$, $\{b_i\}$ and parameters τ , δ . In particular, we will focus on the monotonicity of the eigenvalues as these coefficients and parameters change. To this end, we consider the following two boundary value problems:

$$\Delta(r_{i-1}^{(1)} \Delta y_{i-1}) - b_i^{(1)} y_i + \lambda a_i^{(1)} y_i = 0, \quad 1 \leq i \leq n, \quad (3.1a)$$

$$y_0 - \tau^{(1)} y_1 = y_{n+1} - \delta^{(1)} y_n = 0, \quad (3.1b)$$

and

$$\Delta(r_{i-1}^{(2)} \Delta y_{i-1}) - b_i^{(2)} y_i + \lambda a_i^{(2)} y_i = 0, \quad 1 \leq i \leq n, \quad (3.2a)$$

$$y_0 - \tau^{(2)} y_1 = y_{n+1} - \delta^{(2)} y_n = 0. \quad (3.2b)$$

For each $t = 1, 2$, we define the matrices $G^{(t)}$, $D^{(t)}$, $A^{(t)}$, and $B^{(t)}$ the same way as before. For example, we define

$$A^{(t)} = \text{diag}(a_1^{(t)}, a_2^{(t)}, \dots, a_{n-1}^{(t)}, a_n^{(t)}), \quad t = 1, 2,$$

$$B^{(t)} = \text{diag}(b_1^{(t)}, b_2^{(t)}, \dots, b_{n-1}^{(t)}, b_n^{(t)}), \quad t = 1, 2,$$

and we still have the relation $G^{(t)} = D^{(t)} + B^{(t)}$ for $t = 1, 2$. For each $t = 1, 2$, we assume that

(H_t) The $\tau^{(t)}$ and $\delta^{(t)}$ are constants in $[0, 1]$ with $\tau^{(t)} + \delta^{(t)} \neq 2$. The $\{r_i^{(t)}\}_{i=0}^n$, $\{a_i^{(t)}\}_{i=1}^n$, and $\{b_i^{(t)}\}_{i=1}^n$ are finite sequences of real numbers such that $r_i^{(t)} > 0$ for $0 \leq i \leq n$ and $b_i^{(t)} \geq 0$ for $1 \leq i \leq n$. And there is at least one nonzero member of the sequence $\{a_i^{(t)}\}_{i=1}^n$.

It is obvious that, under the hypotheses of (H₁) and (H₂), both $G^{(t)}$, $t = 1, 2$ are positive definite as pointed out in Section 2.

Theorem 3.1. Assume the hypotheses of (H₁) and (H₂). Let $\tau^{(1)} = \tau^{(2)}$, $\delta^{(1)} = \delta^{(2)}$ and $r_i^{(1)} = r_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, for each i . Let p_t and q_t be the number of positive and number of negative elements in the set $\{a_1^{(t)}, a_2^{(t)}, \dots, a_n^{(t)}\}$ for $t = 1, 2$ and let

$$\{\lambda_{q_t}^-(t) < \dots < \lambda_2^-(t) < \lambda_1^-(t)\} \quad \text{and} \quad \{\lambda_1^+(t) < \lambda_2^+(t) < \dots < \lambda_{p_t}^+(t)\}$$

be the set of all the negative and the set of all the positive eigenvalues of problems (3.1) and (3.2), respectively. If $a_i^{(1)} \geq a_i^{(2)}$ for $1 \leq i \leq n$, then

$$\lambda_k^+(1) \leq \lambda_k^+(2), \quad 1 \leq k \leq p_2 \quad \text{and} \quad \lambda_k^-(1) \leq \lambda_k^-(2), \quad 1 \leq k \leq q_1. \quad (3.3)$$

If $a_i^{(1)} > a_i^{(2)}$, $1 \leq i \leq n$, then all the inequalities of (3.3) are strict.

Proof. Under the hypotheses of Theorem 3.1, we have $G^{(1)} = G^{(2)}$. Let LL^T be the Cholesky decomposition of $G^{(1)}$. Define

$$\alpha_k^+ = \frac{1}{\lambda_k^+(1)}, \quad 1 \leq k \leq p_1, \quad \alpha_k^- = \frac{1}{\lambda_k^-(1)}, \quad 1 \leq k \leq q_1, \quad (3.4)$$

$$\beta_k^+ = \frac{1}{\lambda_k^+(2)}, \quad 1 \leq k \leq p_2, \quad \beta_k^- = \frac{1}{\lambda_k^-(2)}, \quad 1 \leq k \leq q_2. \quad (3.5)$$

In view of Theorem 2.4, by inserting $n - (p_1 + q_1)$ zeros in (3.6) and $n - (p_2 + q_2)$ zeros in (3.7), we deduce that

$$\alpha_1^+ > \alpha_2^+ \cdots > \alpha_{p_1}^+ > 0 = \cdots = 0 > \alpha_{q_1}^- > \cdots > \alpha_2^- > \alpha_1^- \quad (3.6)$$

and

$$\beta_1^+ > \beta_2^+ \cdots > \beta_{p_2}^+ > 0 = \cdots = 0 > \beta_{q_2}^- > \cdots > \beta_2^- > \beta_1^- \quad (3.7)$$

are all the eigenvalues of $L^{-1}A^{(t)}L^{-T}$ for $t = 1, 2$, respectively. If $a_i^{(1)} \geq a_i^{(2)}$ for $1 \leq i \leq n$, then $p_2 \leq p_1$ and $q_1 \leq q_2$. Furthermore, $A^{(1)} - A^{(2)}$ is positive semidefinite and so is $L^{-1}A^{(1)}L^{-T} - L^{-1}A^{(2)}L^{-T}$. If $a_i^{(1)} > a_i^{(2)}$ for $1 \leq i \leq n$, then $A^{(1)} - A^{(2)}$ is positive definite and so is $L^{-1}A^{(1)}L^{-T} - L^{-1}A^{(2)}L^{-T}$. It is seen from the monotonic behavior of eigenvalues of symmetric matrices [2, Theorem 3, p. 117] that $\lambda_k(L^{-1}A^{(1)}L^{-T}) \geq \lambda_k(L^{-1}A^{(2)}L^{-T})$ for each k if $a_i^{(1)} \geq a_i^{(2)}$, $1 \leq i \leq n$ and that $\lambda_k(L^{-1}A^{(1)}L^{-T}) > \lambda_k(L^{-1}A^{(2)}L^{-T})$ for each k if $a_i^{(1)} > a_i^{(2)}$, $1 \leq i \leq n$. Thus, the desired results follow immediately from (3.4)–(3.7). \square

In the previous theorem, we studied the monotonic behavior of the eigenvalues of the problem as the sequence $\{a_i\}$ changes while the other parameters of the problem are kept the same. Finally, we will present a result on eigenvalue comparisons which allows a simultaneous change of all coefficients and parameters. To this end, we need to recall the following observation made in [12]: for $t = 1, 2$, if $\delta^{(t)} = 1$, then

$$P_2 P_3 \cdots P_n D^{(t)} P_n^T \cdots P_2^T = \text{diag} \left((1 - \tau^{(t)})r_0^{(t)}, r_1^{(t)}, \dots, r_{n-1}^{(t)} \right),$$

where $P_k = I + e_{k-1}e_k^T$ and e_k is the k th column of the identity matrix I of order n . Thus, for a general $\delta^{(t)}$, we have

$$D^{(t)} = C \text{diag} \left((1 - \tau^{(t)})r_0^{(t)}, \dots, r_{n-1}^{(t)} \right) C^T + e_n(1 - \delta^{(t)})r_n^{(t)}e_n^T, \quad (3.8)$$

where $C = (\prod_{k=2}^n P_k)^{-1}$. The representation of $D^{(t)}$ in (3.8) reveals its structure through which we can easily obtain many of its useful properties.

For any n -dimensional real vector x , define $z = C^T x$, and we deduce from (3.8) that

$$\begin{aligned} x^T(G^{(2)} - G^{(1)})x &= \left((1 - \tau^{(2)})r_0^{(2)} - (1 - \tau^{(1)})r_0^{(1)} \right) z_1^2 + \sum_{i=2}^n (r_{i-1}^{(2)} - r_{i-1}^{(1)}) z_i^2 \\ &\quad + \left((1 - \delta^{(2)})r_n^{(2)} - (1 - \delta^{(1)})r_n^{(1)} \right) (e_n^T x)^2 \\ &\quad + \sum_{i=1}^n (b_i^{(2)} - b_i^{(1)}) x_i^2. \end{aligned} \quad (3.9)$$

Lemma 3.2. Assume the hypotheses of (H_1) and (H_2) . If

$$\left. \begin{aligned} 0 \leq \tau^{(2)} \leq \tau^{(1)} \leq 1, \quad 0 \leq \delta^{(2)} \leq \delta^{(1)} \leq 1, \\ 0 < r_i^{(1)} \leq r_i^{(2)}, \quad 0 \leq b_i^{(1)} \leq b_i^{(2)}, \quad \text{for each } i, \end{aligned} \right\} \quad (3.10)$$

then $(G^{(2)})^{-1} \leq (G^{(1)})^{-1}$. Under the assumptions in (3.10), if we further assume that

$$\tau^{(2)} < \tau^{(1)}, \quad r_i^{(1)} < r_i^{(2)} \quad \text{for all } i, \quad (3.11)$$

or assume that

$$\delta^{(2)} < \delta^{(1)}, \quad r_i^{(1)} < r_i^{(2)} \quad \text{for all } i, \quad (3.12)$$

or assume that

$$b_i^{(1)} < b_i^{(2)} \quad \text{for all } i, \quad (3.13)$$

then $(G^{(2)})^{-1} < (G^{(1)})^{-1}$.

Proof. It is easily seen from (3.10) that

$$(1 - \tau^{(2)})r_0^{(2)} \geq (1 - \tau^{(1)})r_0^{(2)} \geq (1 - \tau^{(1)})r_0^{(1)}, \quad (3.14)$$

and

$$(1 - \delta^{(2)})r_n^{(2)} \geq (1 - \delta^{(1)})r_n^{(2)} \geq (1 - \delta^{(1)})r_n^{(1)}. \quad (3.15)$$

Combining equations from (3.9) to (3.15), we see that $x^T(G^{(2)} - G^{(1)})x \geq 0$ for any n -dimensional real vector x . Therefore, we have $G^{(2)} \geq G^{(1)} > 0$.

Furthermore, if $x \neq 0$, then so is z since $z = C^T x$ and C is nonsingular. From (3.10) and (3.11) the first inequality of (3.14) is strict and thus the coefficient of z_i of (3.9) is positive for each i and other terms are nonnegative. Similarly, from (3.10) and (3.13) the coefficient of x_i is positive for each i and the other terms are nonnegative. Therefore, $x^T(G^{(2)} - G^{(1)})x > 0$ for $x \neq 0$, i.e., $G^{(2)} > G^{(1)} > 0$ for both cases.

From (3.10) and (3.12), the first inequality of (3.15) is strict and thus the coefficients of $(e_n^T x)^2$ and z_i of (3.9) are positive for $2 \leq i \leq n$ and other terms are nonnegative. In this case, whenever $x^T(G^{(2)} - G^{(1)})x = 0$, we have $e_n^T x = 0$ and $z_i = 0$, $2 \leq i \leq n$. Notice that

$$x = C^{-T}z = P_n^T P_{n-1}^T \cdots P_3^T P_2^T (z_1, 0, \dots, 0)^T = (z_1, z_1, \dots, z_1)^T,$$

implying $z_1 = e_n^T (z_1, z_1, \dots, z_1)^T = e_n^T x = 0$. Thus, z is zero and so is x . Therefore, we also have $x^T(G^{(2)} - G^{(1)})x > 0$ for $x \neq 0$, i.e., $G^{(2)} > G^{(1)} > 0$ for this case.

Finally, the desired results follow directly from the monotonicity of the inverse of positive definite matrices [14, Theorem 24, p. 22]. \square

Theorem 3.3. Assume the hypotheses of (H_1) and (H_2) . Assume that the inequalities in (3.10) and

$$a_i^{(1)} \geq a_i^{(2)} \geq 0 \quad \text{for all } i, \quad (3.16)$$

are satisfied. Let p_t be the number of positive elements in the set $\{a_1^{(t)}, a_2^{(t)}, \dots, a_n^{(t)}\}$ for $t = 1, 2$ and let

$$\{\lambda_1^+(t) < \lambda_2^+(t) < \cdots < \lambda_{p_t}^+(t)\}$$

be the set of all eigenvalues of problems (3.1) and (3.2), respectively. Then

$$\lambda_k^+(1) \leq \lambda_k^+(2), \quad 1 \leq k \leq p_2. \quad (3.17)$$

Furthermore, if either

$$(i) \ a_i^{(1)} > a_i^{(2)}, \ 1 \leq i \leq n, \text{ or}$$

(ii) $\alpha_i^{(1)} > 0$ for all i , and one of the three conditions (3.11)–(3.13)

is satisfied, then all the inequalities of (3.17) are strict.

Proof. Obviously, $p_1 \geq p_2$. Let $L_t L_t^T = G^{(t)}$ be the Cholesky decomposition of $G^{(t)}$ for $t = 1, 2$. In view of Theorem 2.4, we have

$$\alpha_1^{(t)} = \frac{1}{\lambda_1^+(t)} > \alpha_2^{(t)} = \frac{1}{\lambda_2^+(t)} > \cdots > \alpha_{p_t}^{(t)} = \frac{1}{\lambda_{p_t}^+(t)} > 0 \quad \text{and} \quad \alpha_{p_t+1}^{(t)} = \cdots = \alpha_n^{(t)} = 0 \quad (3.18)$$

are the eigenvalues of $L_t^{-1} A^{(t)} L_t^{-T}$, $t = 1, 2$. If (3.16) is satisfied, then $A^{(1)} \geq A^{(2)} \geq 0$ which implies

$$L_2^{-1} A^{(1)} L_2^{-T} \geq L_2^{-1} A^{(2)} L_2^{-T}. \quad (3.19)$$

Denote by $\sqrt{A^{(1)}}$ the diagonal matrix whose diagonal elements are $\left\{ \sqrt{a_1^{(1)}}, \dots, \sqrt{a_n^{(1)}} \right\}$. It is seen from Lemma 3.2 that

$$\sqrt{A^{(1)}} (G^{(2)})^{-1} \sqrt{A^{(1)}} \leq \sqrt{A^{(1)}} (G^{(1)})^{-1} \sqrt{A^{(1)}}. \quad (3.20)$$

Eqs. (3.19) and (3.20), together with the monotonic behavior of eigenvalues of symmetric matrices [2, Theorem 3, p. 117], lead to

$$\lambda_k \left(L_2^{-1} A^{(2)} L_2^{-T} \right) \leq \lambda_k \left(L_2^{-1} A^{(1)} L_2^{-T} \right) \quad \text{for each } k, \quad (3.21)$$

and

$$\lambda_k \left(\sqrt{A^{(1)}} (G^{(2)})^{-1} \sqrt{A^{(1)}} \right) \leq \lambda_k \left(\sqrt{A^{(1)}} (G^{(1)})^{-1} \sqrt{A^{(1)}} \right) \quad \text{for each } k. \quad (3.22)$$

It is well known that (see [14, Theorem 9, p. 14])

$$\lambda_k \left(L_2^{-1} A^{(1)} L_2^{-T} \right) = \lambda_k \left(\sqrt{A^{(1)}} L_2^{-T} L_2^{-1} \sqrt{A^{(1)}} \right) = \lambda_k \left(\sqrt{A^{(1)}} (G^{(2)})^{-1} \sqrt{A^{(1)}} \right) \quad (3.23)$$

and

$$\lambda_k \left(L_1^{-1} A^{(1)} L_1^{-T} \right) = \lambda_k \left(\sqrt{A^{(1)}} L_1^{-T} L_1^{-1} \sqrt{A^{(1)}} \right) = \lambda_k \left(\sqrt{A^{(1)}} (G^{(1)})^{-1} \sqrt{A^{(1)}} \right). \quad (3.24)$$

Combining (3.21)–(3.24), we can write

$$\begin{aligned} \alpha_k^{(2)} &= \lambda_k \left(L_2^{-1} A^{(2)} L_2^{-T} \right) \leq \lambda_k \left(L_2^{-1} A^{(1)} L_2^{-T} \right) \\ &= \lambda_k \left(\sqrt{A^{(1)}} (G^{(2)})^{-1} \sqrt{A^{(1)}} \right) \\ &\leq \lambda_k \left(\sqrt{A^{(1)}} (G^{(1)})^{-1} \sqrt{A^{(1)}} \right) \\ &= \lambda_k \left(L_1^{-1} A^{(1)} L_1^{-T} \right) = \alpha_k^{(1)}, \quad 1 \leq k \leq n. \end{aligned} \quad (3.25)$$

Therefore, the inequalities of (3.17) follow from (3.18) and (3.25).

Moreover, if (i) is satisfied, then the inequality of (3.19) is strict and so are those of (3.21). If (ii) is satisfied, then, in view of Lemma 3.2 we have

$$A^{(1)} > 0, \quad (G^{(2)})^{-1} < (G^{(1)})^{-1}.$$

Hence, the inequality of (3.20) is strict and so are (3.22). Therefore, $\alpha_k^{(2)} < \alpha_k^{(1)}$ for each k in all cases, i.e., the inequalities of (3.17) are strict under the conditions in either (i) or (ii). \square

References

- [1] F.V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York, 1964.
- [2] R. Bellman, *Introduction to Matrix Analysis*, second ed., SIAM, Philadelphia, PA, 1997.
- [3] J.M. Davis, P.W. Eloe, J. Henderson, Comparison of eigenvalues for discrete Lidstone boundary value problems, *Dynam. Systems Appl.* 8 (3–4) (1999) 381–388.
- [4] J. Diaz, A. Peterson, Comparison theorems for a right disfocal eigenvalue problem, *Inequalities and Applications*, World Sci. Ser. Appl. Anal., vol. 3, World Sci. Publishing, River Edge, NJ, 1994, pp. 149–177.
- [5] F.R. Gantmacher, *The Theory of Matrices*, vol. 1, Chelsea Publishing Company, New York, NY, 1960.
- [6] D. Hankerson, J. Henderson, Comparison of eigenvalues for n -point boundary value problems for difference equations, *Differential Equations* (Colorado Springs, CO, 1989), *Lecture Notes in Pure and Appl. Math.*, vol. 127, Dekker, New York, 1991, pp. 203–208.
- [7] D. Hankerson, A. Peterson, Comparison of eigenvalues for focal point problems for n th order difference equations, *Differential Integral Equations* 3 (2) (1990) 363–380.
- [8] D. Hankerson, A. Peterson, A positivity result applied to difference equations, *J. Approx. Theory* 59 (1989) 76–86.
- [9] D. Hankerson, A. Peterson, Comparison theorems for eigenvalue problems for n th order differential equations, *Proc. Amer. Math. Soc.* 104 (4) (1988) 1204–1211.
- [10] J. Henderson, K.R. Prasad, Comparison of eigenvalues for Lidstone boundary value problems on a measure chain, *Comput. Math. Appl.* 38 (11–12) (1999) 55–62.
- [11] A. Jirari, Second-order Sturm–Liouville difference equations and orthogonal polynomials, *Mem. Amer. Math. Soc.* 113 (1995).
- [12] J. Ji, B. Yang, Eigenvalue comparisons for boundary value problems for second order difference equations, *J. Math. Anal. Appl.* 320 (2) (2006) 964–972.
- [13] E.R. Kaufmann, Comparison of eigenvalues for eigenvalue problems of a right disfocal operator, *Pan. Amer. Math. J.* 4 (4) (1994) 103–124.
- [14] J.R. Magnus, H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, John Wiley & Sons Ltd., New York, NY, 1988.
- [15] Y. Shi, S. Chen, Spectral theory of second-order vector difference equations, *J. Math. Anal. Appl.* 239 (2) (1999) 195–212.
- [16] Y. Shi, S. Chen, Spectral theory of higher-order discrete vector Sturm–Liouville problems, *Linear Algebra Appl.* 323 (1–3) (2001) 7–36.
- [17] C.C. Travis, Comparison of eigenvalues for linear differential equations of order $2n$, *Trans. Amer. Math. Soc.* 177 (1973) 363–374.